Universidad Simón Bolívar
Departamento de Matemáticas
Puras y Aplicadas
Enero–Marzo 00

Nombre: ______________________

Carnet: ____________ Sección: ____

MA2112. 1er. Parcial. 7:30 AM. B

1. (14 ptos.) Hallar y clasificar los puntos críticos de $f(x,y) = x^4 + \frac{2}{3}y^3 + 4y^2 - x^2y^2 + 2$.

Puntos críticos $\to \nabla f(x,y) = (0,0)$.

$\frac{\partial f}{\partial x}(x,y) = 4x^3 - 2xy^2 = 0 \Rightarrow 2x(2x^2 - y^2) = 0 \Rightarrow y^2 = 2x^2 (1) \wedge x = 0 (2)$

$\frac{\partial f}{\partial y}(x,y) = 2y^2 + 8y - 2x^2y = 0 \Rightarrow 2y(2y + 8 - 2x^2) = 0 \Rightarrow 2y + 8 = 2x^2 (3) \wedge y = 0 (4)$

(1) con (4): $x = 0, y = 0 \Rightarrow P_1 (0,0)$.

(1) con (3): $y^2 = 2y + 8 \Rightarrow y^2 - 2y - 8 = 0 \Rightarrow y = \frac{2 \pm \sqrt{4 - 4(1)(-8)}}{2} = \frac{2 \pm \sqrt{36}}{2}$

$= \frac{2 \pm 6}{2} \Rightarrow y_1 = 4 \wedge y_2 = -2. \Rightarrow$ Si $y = 4 \Rightarrow 16 = 2x^2 \Rightarrow x = \pm\sqrt{8}$

Si $y = -2 \Rightarrow 4 = 2x^2 \Rightarrow x = \pm\sqrt{2}$.

$P_2(\sqrt{8}, 4)$, $P_3(-\sqrt{8}, 4)$, $P_4(\sqrt{2}, -2)$, $P_5(-\sqrt{2}, -2)$

(3) con (4): $2(0) + 8 = 2x^2 \Rightarrow 8 = 2x^2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$, $P_6(2,0)$, $P_7(-2,0)$

(2) con (3): $2y + 8 = 0 \Rightarrow y = -4$, $P_8(0,-4)$

Hallamos las segundas derivadas:

$\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2 - 2y^2$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 4y + 8 - 2x^2$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -4xy$

Clasifiquemos:

Punto	$\frac{\partial^2 f}{\partial x^2}$	$\frac{\partial^2 f}{\partial y^2}$	$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$	$H(x,y)$	Conclusión
$(0,0)$	0	8	0	0	No concluye.
$(\sqrt{8},4)$	+	+	+	+	mínimo.
$(-\sqrt{8},4)$					

2. (13 ptos.) Sean $F : \mathbb{R}^2 \to \mathbb{R}$ de clase C^1, $u(x, y, z) = x \cos y$ y $v(x, y, z) = 2x - z$. Sabiendo que la ecuación del plano tangente a la superficie de ecuación $G(x, y, z) = F(u(x, y, z), v(x, y, z)) = 4$ en el punto $(1, 0, 2)$ es $3x - z = 1$; hallar la ecuación de la recta tangente a $F(u, v) = 4$ en el punto $(1, 0)$.

$G(x,y,z) = F(u(x,y,z), v(x,y,z)) = F(k(x,y,z))$, donde $k(x,y,z) = \begin{pmatrix} x\cos(y) \\ 2x-z \end{pmatrix}$

$\nabla G(1,0,2) = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \nabla G(x,y,z) = \nabla F(k(x,y,z))\, Dk(x,y,z)$

Pero, $\nabla G(1,0,2) = \nabla F(k(1,0,2))\, \nabla k(1,0,2)$. Sin embargo:

$k(1,0,2) = \begin{pmatrix} u(1,0,2) \\ v(1,0,2) \end{pmatrix} = \begin{pmatrix} (1)\cos(0) \\ 2(1)-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$Dk(1,0,2) = \begin{pmatrix} \cos(y) & -x\operatorname{sen}(y) & 0 \\ 2 & 0 & -1 \end{pmatrix}\Big|_{(1,0,2)} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

Entonces:

$(3,0,-1) = \nabla F(1,0) \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix} \Rightarrow (3,0,-1) = \left(\frac{\partial F}{\partial u}(1,0) \quad \frac{\partial F}{\partial v}(1,0)\right) a.$

$\Rightarrow (3,0,-1) = (F_u \quad F_v) \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix} = (F_u + 2F_v,\ 0,\ -F_v)$

obteniendo que $-F_v = -1 \Rightarrow F_v = 1 = \frac{\partial F}{\partial v}(1,0)$ y que $F_u = 3-2 = 1 = \frac{\partial F}{\partial u}(1,0)$

$\Rightarrow \nabla F(1,0) = (1,1)$

Como $F : \mathbb{R}^2 \to \mathbb{R}$ y $w = F(u,v) \Rightarrow F(u,v) - w = 0 = f(u,v,w)$

$\nabla f(u,v,w) = (1,1,-1)$

$\Rightarrow \frac{\partial F}{\partial u}(1,0)(u-1) + \frac{\partial F}{\partial v}(1,0)(v-0) = 0$

$\Rightarrow u - 1 + v = 0 \Rightarrow u + v = 1.$

3. (13 ptos.) La ecuación $u + e^u = x^2y$ define implícitamente a la función $u = f(x, y)$. Suponiendo que f es C^3, escribir el polinomio de Taylor de grado 2 de f entorno a $(1, 1)$.

$$f(x,y) \approx f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(1,1)(x-1)^2 + \frac{\partial^2 f}{\partial x \partial y}(1,1)(x-1)(y-1) + \frac{\partial^2 f}{\partial y^2}(1,1)(y-1)^2\right)$$

Derivamos con respecto a x:

$$\frac{\partial u}{\partial x} + e^u \frac{\partial u}{\partial x} = 2xy \Rightarrow \frac{\partial u}{\partial x} = \frac{2xy}{1+e^u} \Rightarrow \frac{\partial u}{\partial x}(1,1) = \frac{2}{1+e^0} = 1.$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{2y(1+e^u) - \left(e^u \frac{\partial u}{\partial x}\right)(2xy)}{(1+e^u)^2} \Rightarrow \frac{\partial^2 u}{\partial x^2}(1,1) = \frac{4-2}{4} = \frac{1}{2}.$$

Derivamos con respecto a y:

$$\frac{\partial u}{\partial y} + e^u \frac{\partial u}{\partial y} = x^2 \Rightarrow \frac{\partial u}{\partial y} = \frac{x^2}{1+e^u} \Rightarrow \frac{\partial u}{\partial y}(1,1) = \frac{1}{2}.$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{x^2\left(e^u \frac{\partial u}{\partial y}\right)}{(1+e^u)^2} = -\frac{\frac{1}{2}}{4} = -\frac{1}{8}.$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{2xy}{1+e^u}\right) = \frac{2x(1+e^u) - \left(e^u \frac{\partial u}{\partial y}\right)(2xy)}{(1+e^u)^2} \Rightarrow \frac{\partial^2 u}{\partial x \partial y}(1,1) = \frac{4-2}{4} = \frac{1}{2}.$$

$$\Rightarrow f(x,y) \approx f(1,1) + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{2}\left(-\frac{1}{8}(y-1)^2 + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)(y-1)\right)$$

4. (10 ptos.) Sea $f : \mathbb{R}^2 \to \mathbb{R}$ dada por

$$f(x,y) = \begin{cases} x^3 \operatorname{sen}(1/x^2) + y^2 & \text{si } x \neq 0 \\ y^2 & \text{si } x = 0. \end{cases}$$

¿Es f diferenciable en todo $\mathbb{R}^2$? **(Justifique su respuesta).**

Sean las funciones: $g(x) = \begin{cases} x^3 \operatorname{sen}\left(\frac{1}{x^2}\right), & \text{si } x \neq 0 \\ 0, & \text{si } x = 0 \end{cases}$, $h(y) = y^2$

Vemos que $f(x,y) = g(x) + h(y)$

Como g es diferenciable y h también $\Rightarrow$ f es diferenciable.

Sea $(x,y) \in \mathbb{R}^2$ tal que $(x,y) = (0,y_0)$, con $y_0 \in \mathbb{R}$

$f(0,y_0) = y_0^2$, $\lim\limits_{(x,y)\to(0,y_0)} x^3 \operatorname{sen}\left(\frac{1}{x^2}\right) + y^2$ Rectas: $y = mx + y_0$

$= \lim\limits_{x\to 0} \left(x^3 \operatorname{sen}\left(\frac{1}{x^2}\right) + (mx+y_0)^2\right) = \lim\limits_{x\to 0} \cancel{\left(x^3 \operatorname{sen}\left(\frac{1}{x^2}\right)\right)}^{0} + \lim\limits_{x\to 0} (mx+y_0)^2$

$= y_0^2 \Rightarrow f$ es continua en $\mathbb{R}^2$